

Reply II to “Comments on nonlinear viscosity and Grad’s moment method”

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The Comments by Santos [Phys. Rev. E **67**, 053201 (2003)] are addressed in this paper. It is shown that his comments are based on an assumption that is not made in the paper [Phys. Rev. E **65**, 031202 (2002)] commented upon by him. It is also shown that the hydrodynamic equations used by him are not the same as those implied by the constitutive equations for the stress tensor elements in the paper under comment. Therefore, the deductions and the conclusion drawn by him are not applicable to the aforementioned stress tensors. His comments on the thermodynamic consistency and the dimensionality of the kinetic equation are also shown to be baseless.

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At the core of the Comments by Santos [1] lies the question of whether or not the approximate evolution equations for macroscopic field variables (i.e., hydrodynamic equations or moment equations) derived from a kinetic equation (e.g., the Boltzmann kinetic equation) are self-consistent from the standpoint of hydrodynamics. Since a one-dimensional kinetic equation is used in his analysis of the question it is appropriate to discuss first the nature of the one-dimensional kinetic equation from the standpoint of kinetic theory of matter. Then we examine the basic ideas and methodologies employed in the kinetic theory for the purpose of deriving the aforementioned evolution equations, before the replies and responses are specifically made to his comments.

A one-dimensional kinetic equation is a mathematical construct that is often used for heuristic reasons to provide us insights into more realistic but often mathematically intractable three-dimensional kinetic equation which is considered to be the molecular theory basis of macroscopic phenomena of interest. It is difficult to believe that a group of molecules are all confined to a line and move one dimensionally in unison, giving rise to a one-dimensional macroscopic fluid motion, as is suggested by the one-dimensional kinetic equation. One-dimensional kinetic equations are against the basic concept of random molecular motions inherent to statistical mechanics, in which macroscopic observables are calculated as mean values of the corresponding molecular expressions of dynamical quantities defined in the multidimensional phase space of many particles. However, they are used often in kinetic theory investigations because they sometimes provide exact solutions and thus invaluable insights into more realistic but complex problems for which exact or analytic solutions are difficult to obtain. Deductions made from them for macroscopic phenomena should be used with caution. Santos seems to think otherwise about them.

It is useful to put the gist of the Reply in a paragraph before presenting the details of the Reply and response to his comments. The mass, momentum, and energy conservation laws (balance equations), Eqs. (4)–(6) of Ref. [1], used by him are one dimensional and do not contain fluid dynamic variables in other than the x direction in the coordinate system. In other words, they do not involve velocity components, stress tensor components, and heat fluxes in the directions of y and z . Therefore the flow is not only unidirectional

but also one dimensional. On the other hand, the unidirectional channel flow underlying the stress tensor calculated in Ref. [2] is not one dimensional in the sense meant by Santos with Eqs. (4)–(6), because flow velocity component u_x and other flow variables depend on x , y , z in general and, in a special case, on both x and y . It is evident from Eq. (28) in Ref. [2] that the stress tensor is not one-dimensional. This means that the hydrodynamic equations, namely, mass, momentum, and energy balance equations corresponding to the stress tensor calculated from Eq. (28) of Ref. [2] are not one-dimensional equations as Eqs. (4)–(6) are, as will be shown in the course of the reply given in the following. Therefore, Santos’ principal conclusion deduced from Eqs. (4)–(6) in his Comment paper [1] is not applicable to the hydrodynamics of flow implied by the stress tensor provided by Eq. (28) of Ref. [2]. *It should be emphasized that the stress tensor calculated from Eq. (28) of Ref. [2] by no means implies Assumption (d) of his.*

One of the principal aims in rheology lies in establishing the rheological constitutive equations for the substance of interest in terms of the shear rate and/or the elongation rate and also in terms of external forces if necessary. In other words, the stress tensor components of the substance in hand are calculated in terms of various velocity gradients or pressure gradients under the influence of which the substance flows. The shear and elongation rates are experimental inputs in the rheological constitutive equations, because in rheology they are treated as experimentally given. This raises some subtle questions in the high shear rate regime. Nevertheless, that is how it is done in rheology according to the textbooks on rheology and rheometry [3,4].

For the purpose of obtaining such constitutive equations the methods of statistical mechanics and kinetic theory are often employed unless the approach is phenomenological. In such a kinetic theory derivation of rheological constitutive equations the flow problem under investigation does not require solutions of a full set of hydrodynamic equations, which includes the conservation laws of mass, momentum, and internal energy and the rheological constitutive equations (evolution equations) for the stress tensors and, perhaps, the constitutive equations (evolution equations) for the heat flux, diffusion fluxes, and other relevant nonconserved variables. The basic reason underlying this approach is that the primary role of rheological constitutive equations is in relating the stress tensors to the shear and elongation rates

(velocity gradients), which are experimental inputs, in a manner compatible with the principles of mechanics and hydrodynamics and also of thermodynamics. Hydrodynamic flow calculations are performed in the next stage with the conservation laws together with the rheological constitutive equations so constructed and elucidated through rheological experiments.

The Chapman-Enskog method [5] is one of the major methods established for solving the Boltzmann kinetic equation or equivalent kinetic equations and for calculating the aforementioned constitutive equations. In the Chapman-Enskog method the constitutive equations for the nonconserved variables arise as the solvability conditions for the kinetic equation order by order with respect to the nonuniformity parameter in a series of which the distribution function is expanded. Thus the Euler equations, the Navier-Stokes-Fourier equations, the Burnett equations, and so on are obtained systematically and self-consistently according to the order of the nonuniformity parameter. The constitutive equations for stresses and heat fluxes—namely, the Newtonian law of viscosity and the Fourier law of heat conduction at the first order level of the Chapman-Enskog solution—do not involve time derivatives, and thus may be regarded as steady state approximations of the moment equations arising, for example, in the Grad moment method [6]. Thus derived hydrodynamic equations are self-consistent and provide molecular theory foundations of the classical hydrodynamics, which adequately accounts for diverse flow phenomena.

On the other hand, in the moment method of Maxwell [7] and Grad [6] the moment evolution equations generated from the kinetic equation provide another set of macroscopic equations for field variables, which may be regarded as equivalent to those of the Chapman-Enskog method. They may be systematically solved by expanding the moments in a series of the nonuniformity parameter. Thus obtained hydrodynamic equations are mathematically self-consistent order by order within the series expansion method employed, and the leading order hydrodynamic equations acquired coincide with those derived by the Chapman-Enskog method. For a given flow problem, depending on the degree of departure from equilibrium, one is then to solve such hydrodynamic equations at an order of the nonuniformity parameter, namely, the Euler, Navier-Stokes-Fourier, or Burnett equations, subject to initial and boundary conditions. Again, the hydrodynamic equations thus derived are mutually consistent regardless of whether a steady state assumption is taken or not. This is especially so in the case of constitutive equations for the nonconserved variables appearing in the moment method. If one wishes to avoid using the aforementioned expansion in the nonuniformity parameter for nonconserved variables then the methodology so developed for solution of the moment evolution equations will take us beyond the level of the classical hydrodynamics. The moment evolution equations still remain self-consistent.

The previous four paragraphs describe a general philosophy and backdrop for various kinetic theory efforts to derive constitutive equations for nonconserved variables in the lit-

erature [5,8]. It also underlies the kinetic theory method used for the derivation of the stress tensor evolution equation in Ref. [2].

In Ref. [2] under Comment by Santos [1] the present author has made an analysis of the constitutive equations for the stress tensor components which Uribe and Garcia-Colin [9] reported in this journal. Their derivation of the constitutive equations is within the bounds of the general philosophy taken in the traditional kinetic theory methods mentioned earlier. The authors of Ref. [9] derived the constitutive equations for the stress tensor under the assumption that the transversal velocity gradients vanish (i.e., $\nabla_y u_x = \nabla_z u_x = 0$) in the case of a unidirectional flow along the x axis (i.e., $u_x \neq 0$ in the flow) and made some deductions claimed to be of general nature regarding the material functions (i.e., shear and longitudinal viscosities). The present author in Ref. [2] pointed out that the aforementioned assumption could give rise to inappropriate constitutive equations for the stress tensor components for the unidirectional flow considered. The thermodynamic consistency of the constitutive equations so obtained were also questioned and commented on in the paper [2]. In addition, a general sort of comment was made with regard to the dimensionality of the kinetic equation often taken in the kinetic theory approaches made in the literature [10], in which the kinetic equation (e.g., the Boltzmann equation) is made one dimensional in the configurational space if the hydrodynamic flow is one dimensional.

It was certainly not advocated by this author in Ref. [2] that the constitutive equations for the stress tensor components in question should be used for rheological studies of the material functions of the gases considered. As a matter of fact, the present author has derived and reported on a set of hydrodynamic equations, called the generalized hydrodynamic equations which are not only thermodynamically consistent but also capable of successfully accounting for experimental data on shock wave structures [11], various flow characteristics in rarefied gases [12], and rheological properties [13] of complex liquids such as polymeric liquids. These are the aspects that the moment equations derived by the Grad moment method are not generally able to account for properly. Consequently there is no compelling reason for the present author to suggest Grad's moment equations for applications in the study of flow problems in fluid systems. It is therefore out of ordinary that Santos takes issue with the present author about the object of Comment by the present author, which the latter concluded defective in Ref. [2] and never intended to use for application. If he finds the constitutive equations for stress tensors by Uribe and Garcia-Colin [9] to have defects the comments should have been directed toward Ref. [9], not to the paper making analysis of their constitutive equations and find them defective, unless there is a problem with the analysis in Ref. [2]. However, since he has addressed to Ref. [2] collateral Comments, which I find are unwarranted, I am obliged to respond to his comments.

Santos attributes five assumptions to Refs. [2] and [9]. Since Assumptions a, b, c, and e are present in Refs. [2,9], but do not have a crucial significance to the alleged inconsistency of the hydrodynamic equations, only Assumption d on the spatial uniformity of the pressure tensor needs to be

discussed here. Nowhere in Ref. [2] is the assumption in question made nor is it inferred or indicated, and it is my interpretation that the same holds true for Ref. [9].

First of all, the notation $\mathbf{\Pi}$ for the excess stress tensor used in Ref. [2] has been incorrectly interpreted and confused with the pressure (stress) tensor \mathbf{P} by Santos. It should be emphasized that $\mathbf{\Pi}$ is not the same as \mathbf{P} . It is defined by $\mathbf{\Pi} = \mathbf{P} - p\delta$, where δ denotes the unit second rank tensor. This confusion has unfortunate and undesirable ramifications in his discourse. The following discussion shows that assumption d is not implied by the constitutive equations in Refs. [2] and [9].

In the aforementioned systematic methods of solving the kinetic equation the underlying principle is the functional hypothesis [5,8,14], in which nonconserved variables are assumed to be functionals of the conserved variables, which determine the time and spatial scales for the macroscopic evolution of the system. Because of this functional hypothesis the derivatives of nonconserved variables appear as terms, at least, one order higher than the nonconserved variables themselves in the aforementioned schemes of expansion, if such an expansion method is employed to solve the kinetic equation. Therefore, for example, at a given order of approximation the derivatives of stress tensor \mathbf{P} in the term such as $\mathbf{u} \cdot \nabla P$ appearing in the constitutive equation for the stress tensor does not occur in the momentum balance equation at the same order of the nonuniformity parameter. Note also that $\nabla \mathbf{P}$ in the term $\mathbf{u} \cdot \nabla \mathbf{P}$ is not the same as $\nabla \cdot \mathbf{P}$. Therefore even if $\mathbf{u} \cdot \nabla \mathbf{P}$ is ignored in the constitutive equation for \mathbf{P} because the former is one order higher than \mathbf{u} in the expansion scheme, it does not mean that the $\nabla \cdot \mathbf{P}$ term should be absent in the momentum balance equation. Besides, the approximate constitutive equations derived in Refs. [2,9] depend on position and hence $\nabla \cdot \mathbf{P} \neq 0$ in general and consequently the term $\nabla \cdot \mathbf{P}$ is not absent in the momentum balance equation (momentum conservation law).

Certainly the constitutive equations for the components of tensor $\mathbf{\Pi}$, which are derived by the Grad moment method in Ref. [2], do not suggest in any way that $\mathbf{\Pi}$ is uniform in the configuration space. In fact, assumption d is contradicted by the constitutive equations for the components of $\mathbf{\Pi}$ presented in Ref. [2]; see Eq. (28) and its approximate (iterative) solutions in Ref. [2]. Setting aside the feature questioned about the stress tensor evolution equations of Uribe and Garcia-Colin in Ref. [2], I would like to state that neither do the evolution equations for the components of \mathbf{P} in Ref. [9] suggest assumption (d).

I would like to point out that, as a matter of fact, the momentum balance equations corresponding to the stress tensor $\mathbf{\Pi}$ for the unidirectional flow described in Ref. [2] are as follows:

$$\rho D_t u_x + \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \Pi_{xx} + \frac{\partial}{\partial y} \Pi_{yx} = 0, \quad (1)$$

$$\frac{\partial p}{\partial y} + \frac{\partial}{\partial x} \Pi_{xy} + \frac{\partial}{\partial y} \Pi_{yy} = 0, \quad (2)$$

where D_t is the substantial time derivative, ρ is the mass density, and $D_t u_y = 0$ because $u_y = 0$. This set of momentum

balance equations arises when the flow in the z direction is neutral in the case of a unidirectional flow parallel with the x axis. One may use the solutions of Eq. (28) of Ref. [2] and, for example, iterative solutions thereof for the components of $\mathbf{\Pi}$ in Eqs. (1) and (2). The normal stress differences are defined by $N_1 = \Pi_{xx} - \Pi_{yy}$ and $N_2 = \Pi_{yy} - \Pi_{zz}$. Since the stress tensor $\mathbf{\Pi}$ is traceless symmetric it follows that $\Pi_{xx} = (2N_1 + N_2)/3$, $\Pi_{yy} = (N_2 - N_1)/3$, and $\Pi_{zz} = -(N_1 + 2N_2)/3$. The normal stress differences are of second order with respect to the shear rate; this can be easily verified with the approximate solutions of Eq. (28) of Ref. [2]. Therefore, if the shear rate is small the normal stress differences may be neglected. In such an approximation and in the event the pressure gradient in the y direction vanishes as is the case for creeping flow under a constant pressure gradient in the x direction, we obtain from Eqs. (1) and (2) the momentum balance equations in the form

$$\rho D_t u_x + \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \Pi_{yx} = 0, \quad (3)$$

$$\frac{\partial}{\partial x} \Pi_{xy} = 0. \quad (4)$$

These equations should be compared with Eq. (5) of Ref. [1] in order to see their differences; for example, Eqs. (3) and (4) are two-dimensional partial differential equations whereas Eq. (5) of Ref. [1] is one dimensional.

Equation (4) implies that Π_{xy} is independent of x , that is, $\Pi_{xy} = \Pi_{xy}(y)$. Therefore, if the flow is under a constant pressure gradient in the x direction then the two-dimensional momentum balance equations (1) and (2) are reduced to a single ordinary differential equation in y . Furthermore, if the channel is infinite in length so that u_x is translationally invariant in the x direction and hence $\partial u_x / \partial x = 0$ then $D_t u_x = 0$. In this case Eq. (3) is the equation for the Poiseuille flow in the channel if the Newtonian law of viscosity is used for Π_{yx} , and it gives rise to the well-known parabolic velocity profile that is in the basis of a rheometric method for measuring viscosity. I would like to point out that the Newtonian law of viscosity is contained as the leading order approximation with respect to the shear rate in the solutions of Eq. (28) of Ref. [2] and that for the unidirectional flow and the accompanying stress tensor calculated in Ref. [2] the momentum and energy balance equations are quite different from Eqs. (5) and (6) of Ref. [1], the most important being the dimensionality of the equations.

Therefore, it can be said that the analysis giving rise to Santos' conclusion that the unidirectional flow considered in Ref. [2,9] is inconsistent with the steady state assumption is inapplicable to the stress tensor calculated in Ref. [2]; it applies to the one-dimensional flow model under his assumptions, which is obviously different from the flow described by Eqs. (1) and (2), because assumption (d) of his does not apply to the constitutive equations for the stress tensor components derived in Ref. [2]. Therefore, his conclusion is incorrectly directed and irrelevant to the hydrodynamic equations accompanied by the constitutive equation for $\mathbf{\Pi}$ derived in Ref. [2].

In Ref. [2] an approximate set of constitutive equations for Π_{xx} , Π_{xy} , and Π_{yz} was suggested as the lowest-order iterative solutions of Eq. (28); see Eqs. (28), (32), (33), and (34) of Ref. [2]. It was also suggested in Ref. [2] that more accurate results could be obtained by means of numerical solutions of Eq. (28). Therefore, it is clear that Eq. (33), for example, will have a limited range of validity, and Santos is pointing out the obvious fact that it indeed has limitations. In fact, such implied limitations were precisely the unstated point of deriving such approximate solutions in addition to the point made about the appearance of the shear stress. This author, however, did not, and does not, advocate them for any application to study of flow problems in Ref. [2]. It is anyone's guess why such an incidental formula should be a point of so belabored an argument, in which he takes pain to distinguish the deductions of the constitutive equations in Refs. [2] and [9] in what seems like an attempt to put the latter in a more favorable light.

Equation (16) of Ref. [1] is an approximation of the evolution equation derived from the Boltzmann equation, and its validity is questionable. Therefore, any deduction made with it is equally dubious. One curious aspect of Eq. (17) is that it is about the stress tensor component P_{yy} which has no place in his one-dimensional flow model because his flow model should have only u_x and P_{xx} , but nothing else. This Comment also applies to Eq. (18) of Ref. [1], which is in fact incorrect. It should read

$$D_t P_{ij} + \frac{\partial}{\partial x_k} \psi_{kij} + \frac{\partial u_k}{\partial x_k} P_{ij} + (P_{kj} \delta_{il} + P_{ki} \delta_{jl}) \frac{\partial u_l}{\partial x_k} = -\mu(P_{ij} - p \delta_{ij}), \quad (5)$$

where the Einstein convention is used for repeated subscripts, $\psi_{kij} = \langle m C_k C_i C_j f \rangle$ with C_l ($l=i, j, k$) denoting the l th component of the peculiar velocity, but the rest of notation is the same as in Ref. [1]. In the unidirectional flow, in which $u_y = u_z = 0$, Eq. (5) should read

$$D_t P_{ij} + \frac{\partial}{\partial x_k} \psi_{kij} + \frac{\partial u_x}{\partial x_x} P_{ij} + \left(P_{kj} \frac{\partial u_i}{\partial x_k} + P_{ki} \frac{\partial u_j}{\partial x_k} \right) = -\mu(P_{ij} - p \delta_{ij}), \quad (6)$$

where $\partial u_i / \partial x_k = \delta_{ix} (\partial u_x / \partial x_k)$ and $\partial u_j / \partial x_k = \delta_{jx} (\partial u_x / \partial x_k)$. On comparing this equation with Eq. (18) of Ref. [1] we find that in Eq. (18) ∂_t should read D_t and the term $\partial \psi_{kij} / \partial x_k$ is missing. In this connection note that ψ_{kij} is not necessarily equal to the heat flux and that for the one-dimensional flow model considered in Ref. [1] there are no other components of \mathbf{P} than the x component, namely, $i=j=x$. The results of the analysis based on Eq. (18) therefore seems to be of a dubious quality.

Having performed an analysis with such a faulty equation, Santos concludes that the stress tensor components obtained as approximations for the solutions of Eq. (28) in Ref. [2] yields "unphysical negative values for the diagonal elements of the pressure tensor." However, as a matter of fact, because of the incorrect interpretation of $\mathbf{\Pi}$ used in Ref. [2] as mentioned earlier, Eq. (21) of Ref. [1] is in error. Conse-

quently the conclusion drawn therefrom is rendered dubious. Besides, since P_{yy} is a normal stress in the dimensions excluded in his one-dimensional flow model, Eq. (21), is extraneous to his flow model and thus cannot be regarded as a result consistent with his flow model. In any case, such approximate solutions for the stress tensors have never been advocated for application in Ref. [2]. They have been used to point out the differences arising when the assumption is made with regard to the transversal velocity gradients by Uribe and Garcia-Colin in Ref. [9]. If the conclusion drawn in Ref. [1] is truthful there is all the more reason to be dubious about the constitutive relations obtained in Ref. [9].

The Grad moment method does not provide theoretical foundations for irreversible thermodynamics, which is consistent with the thermodynamics laws at all orders of approximation, and this point was demonstrated in the literature [15,16]. Therefore, the constitutive equations for nonconserved variables obtained by the Grad moment method are generally inconsistent with the thermodynamic laws. This was a point repeated in Ref. [2]. The particular equations derived from the one-dimensional Boltzmann equation and used by Santos [1] may turn out to be thermodynamically consistent, but he has not shown that they are indeed thermodynamically consistent. In this connection it should be clearly understood that the H theorem satisfied by the Boltzmann equation is not the same as the second law of thermodynamics. The H theorem is broader than the second law of thermodynamics, being a stability theorem for the homogenous steady state solution of the Boltzmann equation, and the macroscopic equations derived from the Boltzmann equation do not necessarily satisfy the second law of thermodynamics, as is shown in the literature [15,16]. I believe that Santos is using the term thermodynamic consistency in the sense broader than I have been using in Ref. [2] and in the literature [15–17] on my work on irreversible thermodynamics. In the Comment paper [1] Santos is arguing without proof that his equations are thermodynamically consistent in the sense I have defined the term earlier. It is yet to be shown to be true for everyone to see.

The phase space density function $F(\mathbf{r}, \mathbf{v}, t)$ in Eq. (25) in Ref. [1] is defined in the $6N$ -dimensional phase space, and if Eq. (25) is averaged over the initial distribution $\rho(\Gamma)$ as suggested in Ref. [1] then the singlet distribution function follows

$$f(\mathbf{r}, \mathbf{v}, t) = \langle F(\mathbf{r}, \mathbf{v}, t) \rangle = \sum_{i=1}^N \int d\Gamma_i \rho(\mathbf{r}, \mathbf{v}, \Gamma_i), \quad (7)$$

where

$$\Gamma_i = (\mathbf{r}_1, \mathbf{v}_1; \cdots; \mathbf{r}_{i-1}, \mathbf{v}_{i-1}; \mathbf{r}_{i+1}, \mathbf{v}_{i+1}; \cdots; \mathbf{r}_N, \mathbf{v}_N),$$

$$d\Gamma_i = d\mathbf{r}_1 d\mathbf{v}_1 \cdots d\mathbf{r}_{i-1} d\mathbf{v}_{i-1} d\mathbf{r}_{i+1} d\mathbf{v}_{i+1} \cdots d\mathbf{r}_N d\mathbf{v}_N.$$

Since the particles are identical it follows that

$$f(\mathbf{r}, \mathbf{v}, t) = N \int d\Gamma_1 \rho(\mathbf{r}, \mathbf{v}, \Gamma_1). \quad (8)$$

This distribution function may indeed obey, for example, the three-dimensional field-free Boltzmann equation

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) f(\mathbf{r}, \mathbf{v}, t) &= \int d\mathbf{v}' \int_0^{2\pi} d\varphi \int_0^\infty dbbg[f(\mathbf{r}, \mathbf{v}^*, t) \\ &\quad \times f(\mathbf{r}, \mathbf{v}'^*; t) - f(\mathbf{r}, \mathbf{v}^*, t)f(\mathbf{r}, \mathbf{v}'^*; t)] \\ &\equiv J[ff] \end{aligned} \quad (9)$$

in the standard notation.

In this manner of defining the singlet distribution function $f(\mathbf{r}, \mathbf{v}, t)$ the one-dimensional distribution function, say, $f(x, \mathbf{v}, t)$ for particles confined to a line parallel to the x axis is obtained if Eq. (8) is integrated over y and z ,

$$\begin{aligned} f(x, \mathbf{v}, t) &= \int dy \int dz f(\mathbf{r}, \mathbf{v}, t) \\ &= N \int dy \int dz \int d\Gamma_1 \rho(\mathbf{r}, \mathbf{v}, \Gamma_1) \end{aligned}$$

and, if the singlet distribution function $f(\mathbf{r}, \mathbf{v}, t)$ obeys kinetic equation (9) then the reduced distribution function $f(x, \mathbf{v}, t)$ obviously obeys the one-dimensional reduced kinetic equation

$$\left(\frac{\partial}{\partial t} + v_x \cdot \nabla_x\right) f(x, \mathbf{v}, t) = \int dy \int dz J[ff](x, y, z). \quad (10)$$

However, $f(x, \mathbf{v}, t)$ does not obey the kinetic equation in which the three-dimensional distribution functions in the Boltzmann collision term are replaced by the one-dimensional distribution function $f(x, \mathbf{v}, t)$, namely, the kinetic equation

$$\left(\frac{\partial}{\partial t} + v_x \cdot \nabla_x\right) f(x, \mathbf{v}, t) = J[f(x, \mathbf{v}, t)f(x, \mathbf{v}, t)], \quad (11)$$

as is often done in many kinetic theory approaches to some hydrodynamic flow problems. And it is the case with Ref. [9]. Equations (10) and (11) are not the same unless the collision integral is linear with respect to f . Therefore, such a manner of using the Boltzmann kinetic equation needs caution. This was the essential point of the remark made in connection with the kinetic equation in Ref. [2]. In the case of the Maxwell model the Boltzmann equation superficially becomes linear if the functional integral depending on the distribution function and related to the relaxation time in the kinetic equation is regarded as a parameter independent of the dimension of the system. Thus it appears as if the kinetic equation can be one dimensional, but because the functional integral related to the relaxation time is not one dimensional, if one has started with the three-dimensional Boltzmann kinetic equation, the kinetic equation is only quasi-one dimensional. Therefore, the claim that the one-dimensional kinetic equation has no problem appears to be acceptable in the case of the Maxwell model, if one ignores the fact that the aforementioned functional integral in the kinetic equation still involves a three-dimensional distribution function. A similar comment applies to the case of a linearized Boltzmann equation. However, as is clear from Eqs. (10) and (11), the conclusion holding for linearized Boltzmann equation does not extend to the nonlinear Boltzmann equation, and it will be a misconception that a kinetic equation can be generally one dimensional in the configuration space if the flow problem under consideration is one dimensional. One should remember that a fluid particle (a packet of molecules) contains a large number of molecules, which move three dimensionally in the configuration space of the phase space for the fluid particle. A one-dimensional kinetic equation, such as Eq. (11), is just a mathematical model, which does not faithfully represent the physical reality in the kinetic theory description of macroscopic systems, if the processes require a nonlinear kinetic equation. Therefore, I do not agree with Santos on the matter regarding the one-dimensional kinetic equation on the point of principle.

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